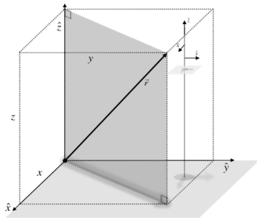
Appendix E Curvilinear Coordinate Systems

Cartesian Coordinates



$$\hat{x}\times\hat{y}=\hat{z}$$

$$\vec{r}=x\,\hat{x}+y\,\hat{y}+z\,\hat{z}$$

$$r=\sqrt{x^2+y^2+z^2}$$

$$\delta\vec{\ell}=\delta x\,\hat{x}+\delta y\,\hat{y}+\delta z\,\hat{z}$$

$$\delta\vec{A}=\delta z\delta y\,\hat{x}+\delta y\delta z\,\hat{y}+\delta x\delta y\,\hat{z}$$

$$\delta V=\delta x\delta z\delta y$$

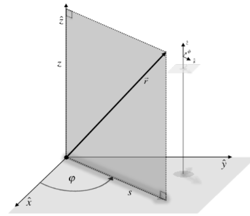
$$\vec{\nabla}\cdot\vec{F}=\tfrac{\partial}{\partial x}F_x+\tfrac{\partial}{\partial y}F_y+\tfrac{\partial}{\partial y}F_z$$

$$\vec{\nabla}f=\tfrac{\partial f}{\partial x}\hat{x}+\tfrac{\partial f}{\partial y}\hat{y}+\tfrac{\partial f}{\partial z}\hat{z}$$

$$\vec{\nabla}\times\vec{F}=\left(\tfrac{\partial}{\partial y}F_z-\tfrac{\partial}{\partial z}F_y\right)\hat{x}+\left(\tfrac{\partial}{\partial z}F_x-\tfrac{\partial}{\partial x}F_z\right)\hat{y}+\left(\tfrac{\partial}{\partial x}F_y-\tfrac{\partial}{\partial y}F_x\right)\hat{z}$$

$$x=s\cos\varphi=r\sin\theta\cos\varphi$$
$$y=s\sin\varphi=r\sin\theta\sin\varphi$$
$$z=z=r\cos\theta$$

Cylindrical Coordinates



$$\hat{s}\times\hat{\varphi}=\hat{z}$$

$$\vec{r}=s\,\hat{s}+z\,\hat{z}$$

$$r=\sqrt{s^2+z^2}$$

$$\delta\vec{\ell}=\delta s\,\hat{s}+s\,\delta\varphi\,\hat{\varphi}+\delta z\,\hat{z}$$

$$\delta\vec{A}=s\,\delta\varphi\,\delta z\,\hat{s}+\delta z\delta s\,\hat{\varphi}+s\,\delta\varphi\,\delta s\,\hat{z}$$

$$\delta V=s\,\delta s\,\delta\varphi\,\delta z$$

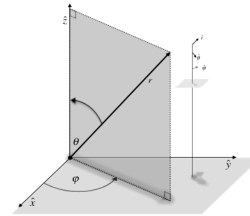
$$\vec{\nabla}\cdot\vec{F}=\tfrac{1}{s}\tfrac{\partial}{\partial s}\left(sF_s\right)+\tfrac{1}{s}\tfrac{\partial}{\partial\varphi}F_\varphi+\tfrac{\partial}{\partial z}F_z$$

$$\vec{\nabla}f=\tfrac{\partial f}{\partial s}\hat{s}+\tfrac{1}{s}\tfrac{\partial f}{\partial\varphi}\hat{\varphi}+\tfrac{\partial f}{\partial z}\hat{z}$$

$$\vec{\nabla}\times\vec{F}=\left(\tfrac{1}{s}\tfrac{\partial}{\partial\varphi}F_z-\tfrac{\partial}{\partial z}F_\varphi\right)\hat{s}+\left(\tfrac{\partial}{\partial z}F_s-\tfrac{\partial}{\partial s}F_z\right)\hat{\varphi}+\left(\tfrac{1}{s}\tfrac{\partial}{\partial s}\left(sF_\varphi\right)-\tfrac{1}{s}\tfrac{\partial}{\partial\varphi}F_s\right)\hat{z}$$

$$s=r\sin\theta=\sqrt{x^2+y^2}$$
$$\varphi=\arctan\left(\tfrac{y}{x}\right)$$
$$z=r\cos\theta$$

Spherical Coordinates



$$\hat{r}\times\hat{\theta}=\hat{\varphi}$$

$$\vec{r}=r\,\hat{r}$$

$$\delta\vec{\ell}=\delta r\,\hat{r}+r\,\delta\theta\,\hat{\theta}+r\sin\theta\,\delta\theta\,\hat{\varphi}$$

$$\delta\vec{A}=r^2\sin\theta\;\delta\theta\,\delta\varphi\,\hat{r}+r\sin\theta\,\delta r\;\delta\varphi\,\hat{\theta}+r\,\delta\theta\,\delta r\,\hat{\varphi}$$

$$\delta V=r^2\,\sin\theta\;\delta r\,\delta\theta\,\delta\varphi$$

$$\vec{\nabla}\cdot\vec{F}=\tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2F_r\right)+\tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\theta}\left(\sin(\theta)F_\theta\right)+\tfrac{1}{r\sin\theta}\tfrac{\partial}{\partial\varphi}F_\varphi$$

$$\vec{\nabla}f=\tfrac{\partial f}{\partial r}\hat{r}+\tfrac{1}{r}\tfrac{\partial f}{\partial\theta}\hat{\theta}+\tfrac{1}{r\sin\theta}\tfrac{\partial f}{\partial\varphi}\hat{\varphi}$$

$$\vec{\nabla}\times\vec{F}=\tfrac{1}{r\sin\theta}\left(\tfrac{\partial}{\partial\theta}\left(\sin\theta\,F_\varphi\right)-\tfrac{\partial}{\partial\varphi}F_\theta\right)\hat{r}+\tfrac{1}{r}\left(\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\varphi}F_r-\tfrac{\partial}{\partial r}\left(rF_\varphi\right)\right)\hat{\theta}+\tfrac{1}{r}\left(\tfrac{\partial}{\partial r}\left(rF_\theta\right)-\tfrac{\partial}{\partial\theta}F_r\right)\hat{\varphi}$$

$$r=\sqrt{x^2+y^2+z^2}=\sqrt{s^2+z^2}$$
$$\theta=\arctan\left(\sqrt{\tfrac{x^2+y^2}{z^2}}\right)=\arctan\left(\tfrac{s}{z}\right)$$
$$\varphi=\arctan\left(\tfrac{y}{x}\right)=\varphi$$

$$\hat{x}=\cos\varphi\,\hat{s}-\sin\varphi\,\hat{\varphi}$$
$$\hat{y}=\sin\varphi\,\hat{s}+\cos\varphi\,\hat{\varphi}$$
$$\hat{z}=\hat{z}$$

$$\hat{s}=\cos\varphi\,\hat{x}+\sin\varphi\,\hat{y}$$
$$\hat{\varphi}=-\sin\varphi\,\hat{x}+\cos\varphi\,\hat{y}$$
$$\hat{z}=\hat{z}$$

$$\hat{s}=\sin\theta\,\hat{r}+\cos\theta\,\hat{\theta}$$
$$\hat{\varphi}=\hat{\varphi}$$
$$\hat{z}=\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}$$

$$\hat{r}=\sin\theta\,\hat{s}+\cos\theta\,\hat{z}$$
$$\hat{\theta}=\cos\theta\,\hat{s}-\sin\theta\,\hat{z}$$
$$\hat{\varphi}=\hat{\varphi}$$

$$\hat{x}=\sin\theta\cos\varphi\,\hat{r}+\cos\theta\,\cos\varphi\,\hat{\theta}-\sin\varphi\hat{\varphi}$$
$$\hat{y}=\sin\theta\sin\varphi\,\hat{r}+\cos\theta\,\sin\varphi\,\hat{\theta}+\cos\varphi\hat{\varphi}$$
$$\hat{z}=\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}$$

$$\hat{r}=\sin\theta\,\cos\varphi\,\hat{x}+\sin\theta\,\sin\varphi\,\hat{y}+\cos\theta\,\hat{z}$$
$$\hat{\theta}=\cos\theta\,\cos\varphi\,\hat{x}+\cos\theta\,\sin\varphi\,\hat{y}-\sin\theta\,\hat{z}$$
$$\hat{\varphi}=-\sin\varphi\,\hat{x}+\cos\varphi\,\hat{y}$$