# Appendix D Vector Calculus

## Geometrical Definitions / Fundamental Theorems

The Divergence

$$\vec{\nabla}\cdot\vec{F} \equiv \lim_{V\to 0}\tfrac{1}{V}\oint_{\text{surface}} \vec{F}\cdot \mathrm{d}\vec{A} \qquad \oint_{\text{surface}} \vec{F}\cdot\mathrm{d}\vec{A} = \oint_{\text{volume}} \left(\vec{\nabla}\cdot\vec{F}\right)\mathrm{d}V$$

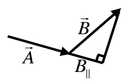
The Gradient

$$\vec{\nabla} f \equiv \lim_{\delta\vec{r}\to 0}\frac{\delta f}{\delta \vec{r}} \qquad f(\vec{r}) - f(\vec{r}_0) = \int_{\vec{r}_0}^{\vec{r}} \vec{\nabla} f\cdot \mathrm{d}\vec{\ell}$$

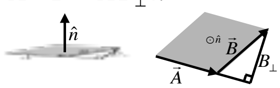
The Curl

$$\left(\vec{\nabla}\times\vec{F}\right)_{\hat{n}} \equiv \lim_{A\to 0}\tfrac{1}{A}\oint_{\text{path}} \vec{F}\cdot\mathrm{d}\vec{\ell}\ \hat{n} \qquad \oint_{\text{path}} \vec{F}\cdot\mathrm{d}\vec{\ell} = \int_{\text{surface}} \left(\vec{\nabla}\times\vec{F}\right)\cdot\mathrm{d}\vec{A}$$

$$\vec{A}\cdot\vec{B} = A\,B_{\|}$$




$$\vec{A}\times\vec{B} = A\,B_{\perp}\,\hat{n}$$




## Vector Algebra Identities

$$A = \left|\vec{A}\right| = \sqrt{\vec{A}\cdot\vec{A}}$$

$$\vec{A}\cdot\vec{A} = A^2$$

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$\vec{A}\times\vec{A} = 0$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}\times\vec{B}\times\vec{C} = \vec{A}\times\left(\vec{B}\times\vec{C}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\times\left(\vec{A}\times\vec{C}\right) - \vec{C}\times\left(\vec{A}\times\vec{B}\right)$$

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) + \vec{B}\times\left(\vec{C}\times\vec{A}\right) + \vec{C}\times\left(\vec{A}\times\vec{B}\right) = 0$$

$$\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right) = \left(\vec{A}\cdot\vec{C}\right)\left(\vec{B}\cdot\vec{D}\right) - \left(\vec{B}\cdot\vec{C}\right)\left(\vec{A}\cdot\vec{D}\right)$$

$$\left(\vec{A}\times\vec{B}\right)\times\left(\vec{C}\times\vec{D}\right) = \left(\vec{A}\cdot\left(\vec{B}\times\vec{D}\right)\right)\vec{C} - \left(\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)\right)\vec{D}$$

## Vector Calculus Identities

$$\vec{\nabla}\times\left(\vec{\nabla} f\right) = 0 \qquad \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$$

$$\nabla^2 f = \vec{\nabla}\cdot\left(\vec{\nabla} f\right) \qquad \nabla^2\vec{A} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$$

$$\vec{\nabla} f(g) = \tfrac{\mathrm{d}f}{\mathrm{d}g}\vec{\nabla} g \qquad \vec{\nabla}\cdot\vec{A}(f) = \vec{\nabla} f\cdot\left(\tfrac{\mathrm{d}}{\mathrm{d}f}\vec{A}\right)$$

$$\nabla^2 f(g) = \tfrac{\mathrm{d}f}{\mathrm{d}g}\nabla^2 g + \tfrac{\mathrm{d}^2 f}{\mathrm{d}g^2}\left|\vec{\nabla} g\right|^2 \qquad \vec{\nabla}\times\vec{A}(f) = \vec{\nabla} f\times\left(\tfrac{\mathrm{d}}{\mathrm{d}f}\vec{A}\right)$$

$$\vec{\nabla}\cdot\left(f\vec{A}\right) = f\left(\vec{\nabla}\cdot\vec{A}\right) + \vec{A}\cdot\left(\vec{\nabla} f\right)$$

$$\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}\times\left(f\vec{A}\right) = f\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\times\left(\vec{\nabla} f\right)$$

$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} - \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)$$

$$\vec{\nabla}\left(f\,g\right) = f\left(\vec{\nabla} g\right) + g\left(\vec{\nabla} f\right)$$

$$\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}$$

$$\nabla^2\left(f\,g\right) = f\left(\nabla^2 g\right) + 2\left(\vec{\nabla} f\right)\cdot\left(\vec{\nabla} g\right) + g\left(\nabla^2 f\right)$$

$$\nabla^2\left(f\vec{A}\right) = f\left(\nabla^2\vec{A}\right) + 2\left(\vec{\nabla} f\cdot\vec{\nabla}\right)\cdot\vec{A} + \vec{A}\left(\nabla^2 f\right)$$

$$\nabla^2\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\cdot\left(\nabla^2\vec{B}\right) - \vec{B}\cdot\left(\nabla^2\vec{A}\right) + 2\vec{\nabla}\cdot\left(\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)\right)$$

### Common Vector Derivatives

$$\vec{\nabla}\cdot\hat{r} = \tfrac{2}{r} \qquad \vec{\nabla}\cdot\vec{r} = 3 \qquad \vec{\nabla}\times\hat{\theta} = \tfrac{\hat{\varphi}}{r}$$

$$\vec{\nabla}\cdot\hat{s} = \tfrac{1}{s} \qquad \vec{\nabla}\times\hat{\varphi} = \tfrac{\hat{z}}{s} = \tfrac{\hat{r}}{r\tan\theta} - \tfrac{\hat{\theta}}{r}$$

$$\vec{\nabla}\cdot\hat{\theta} = \tfrac{1}{r\tan\theta} \qquad \vec{\nabla}\cdot\left(r^n\,\hat{r}\right) = \left(n+2\right)r^{n-1}$$

### Common Curl and Divergence Free Vector Fields

$$\vec{F}\propto\frac{\hat{s}}{s} \qquad \vec{F}\propto\frac{\hat{\varphi}}{s} \qquad \vec{F}\propto\frac{\hat{r}}{r^2} \qquad \vec{F}\propto\frac{3\left(\hat{z}\cdot\hat{r}\right)-\hat{z}}{r^3}$$

$$\vec{\nabla}\times\vec{r} = 0 \qquad \vec{\nabla}\times f(r)\,\hat{r} = 0$$

$$\vec{\nabla} r = \hat{r} \qquad \vec{\nabla} z = \hat{z} \quad \vec{\nabla} s = \hat{s}$$